A critical phenomenon for sublinear elliptic equations in cone–like domains

Vladimir Kondratiev
Department of Mathematics
and Mechanics
Moscow State University
Moscow 119 899, Russia
kondrat@vnmok.math.msu.su

Vitali Liskevich
School of Mathematics
University of Bristol
Bristol BS8 1TW
United Kingdom
v.liskevich@bristol.ac.uk

Vitaly Moroz
School of Mathematics
University of Bristol
Bristol BS8 1TW
United Kingdom
v.moroz@bristol.ac.uk

Zeev Sobol
Department of Mathematics
Imperial College, London
London SW7 2AZ
United Kingdom
z.sobol@imperial.ac.uk

Abstract

We study positive supersolutions to an elliptic equation $(*)$ $-\Delta u = c|x|^{-s}u^p$, $p, s \in \mathbb{R}$, in cone–like domains in $\mathbb{R}^N$ $(N \geq 2)$. We prove that in the sublinear case $p < 1$ there exists a critical exponent $p_* < 1$ such that equation $(*)$ has a positive supersolution if and only if $-\infty < p < p_*$. The value of p_* is determined explicitly by s and the geometry of the cone.

1 Introduction

We study the existence and nonexistence of positive solutions and supersolutions to the equation

$$-\Delta u = \frac{c}{|x|^s}u^p \quad \text{in } \mathcal{C}^\rho_\Omega. \tag{1}$$

Here $p \in \mathbb{R}$, $s \in \mathbb{R}$, $c > 0$ and $\mathcal{C}^\rho_\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an unbounded cone–like domain

$$\mathcal{C}^\rho_\Omega := \{(r, \omega) \in \mathbb{R}^N : \omega \in \Omega,\ r > \rho\},$$

where (r, ω) are the polar coordinates in $\mathbb{R}^N$, $\rho > 0$ and $\Omega \subseteq S^{N-1}$ is a subdomain (a connected open subset) of the unit sphere S^{N-1} in $\mathbb{R}^N$. We say that $u \in H^1_{loc}(\mathcal{C}^\rho_\Omega)$ is a *supersolution* (*subsolution*) to equation (1) if

$$\int_{\mathcal{C}^\rho_\Omega} \nabla u \cdot \nabla\varphi\, dx \geq (\leq) \int_{\mathcal{C}^\rho_\Omega} \frac{c}{|x|^s}u^p\varphi\, dx \quad \text{for all } 0 \leq \varphi \in C^\infty_0(\mathcal{C}^\rho_\Omega).$$

If u is a sub and supersolution to (1) then u is said to be a *solution* to (1). By the weak Harnack inequality any nontrivial nonnegative supersolution to (1) is positive in $\mathcal{C}^\rho_\Omega$.

We define *critical exponents* for equation (1) by

$$p^* = p^*(\Omega, s) = \inf\{p > 1 : (1) \text{ has a positive supersolution in } \mathcal{C}^\rho_\Omega \text{ for some } \rho > 0\},$$

$$p_* = p_*(\Omega, s) = \sup\{p < 1 : (1) \text{ has a positive supersolution in } \mathcal{C}^\rho_\Omega \text{ for some } \rho > 0\}.$$

Set $p_* = -\infty$ if (1) has no positive supersolution in $\mathcal{C}^\rho_\Omega$ for any $p < 1$.

Remark 1. (i) One can show that if $p < p_*$ or $p > p^*$ then (1) has a positive solution in $\mathcal{C}^\rho_\Omega$ (see [6] for the proof of the case $p > 1$ and the proofs below for the case $p < 1$). The existence (or nonexistence) of positive (super) solutions at the critical values p_* and p^* is a separate issue.

(ii) Observe that in view of the scaling invariance of the Laplacian the critical exponents p_* and p^* do not depend on $\rho > 0$.

(iii) We do not make any assumptions on the smoothness of the domain $\Omega \subseteq S^{N-1}$.

Let $\lambda_1 = \lambda_1(\Omega) \geq 0$ be the principal eigenvalue of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ on Ω. Let $\alpha_+ \geq 0$ and $\alpha_- < 0$ be the roots of the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_1(\Omega).$$

In the *superlinear* case $p > 1$ the value of the critical exponent is $p^* = 1 - \frac{2-s}{\alpha_-}$. Moreover, if $s < 2$ then (1) has no positive supersolutions in the critical case $p = p^*$. This has been proved by Bandle and Levine [3], Bandle and Essen [2] and Berestycki, Capuzzo–Dolcetta and Nirenberg [4] (see also [6] for yet another proof of this result and for equations with measurable coefficients).

The *sublinear* case $p < 1$ has been studied in [5, 7]. From the result of Brezis and Kamin [5] it follows that for $p \in (0, 1)$ equation (1) has a bounded positive solution in $\mathbb{R}^N$ if and only if $s > 2$. It has been proved in [7] (amongst other things) that for any $p \in (-\infty, 1)$ equation (1) has a positive supersolution outside a ball in $\mathbb{R}^N$ if and only if $s > 2$.

In this note, we discover a new critical phenomenon. Namely, we show that in sublinear case equation (1) exhibits a "non-trivial" critical exponent ($p_* > -\infty$) in cone-like domains. The main result of the paper reads as follows.

Theorem 1. *For $p \leq 1$, the critical exponent for equation (1) is $p_* = \min\{1 - \frac{2-s}{\alpha_+}, 1\}$. If $p_* < 1$ then (1) has no positive supersolutions in the critical case $p = p_*$.*

Remark 2. (i) If $\alpha_+ = 0$ then we set $p_* = -\infty$.

(ii) If $s > 2$ then $p_* = p^* = 1$ and (1) has positive solutions for any $p \in \mathbb{R}$ [5, 7]. If $s = 2$ then $p_* = p^* = 1$. In this critical case (1) becomes a linear equation with the potential $c|x|^{-2}$, which has a positive (super) solution if and only if $c \leq \frac{(N-2)^2}{4} + \lambda_1(\Omega)$.

(iii) Let $S_k = \{x \in S^{N-1} : x_1 > 0, \dots x_k > 0\}$. Then $\lambda_1(S_k) = k(k + N - 2)$ and $\alpha_+(S_k) = k$, $\alpha_-(S_k) = 2 - N - k$. Hence $p_*(S_k, s) = 1 - \frac{2-s}{k}$ and $p^*(S_k, s) = 1 - \frac{2-s}{2-N-k}$. In particular, in the case of the halfspace S_1 we have $p_*(S_1, s) = s - 1$ and $p^*(S_1, s) = \frac{N+1-s}{N-1}$.

Applying the Kelvin transformation $y = y(x) = \frac{x}{|x|^2}$ we see that if u is a positive solution to (1) in $\mathcal{C}^1_\Omega$ then $\hat{u}(y) = |y|^{2-N} u(x(y))$ is a positive solution to

$$-\Delta \hat{u} = \frac{c}{|y|^\sigma} \hat{u}^p \quad \text{in } \widehat{\mathcal{C}}^1_\Omega, \tag{2}$$

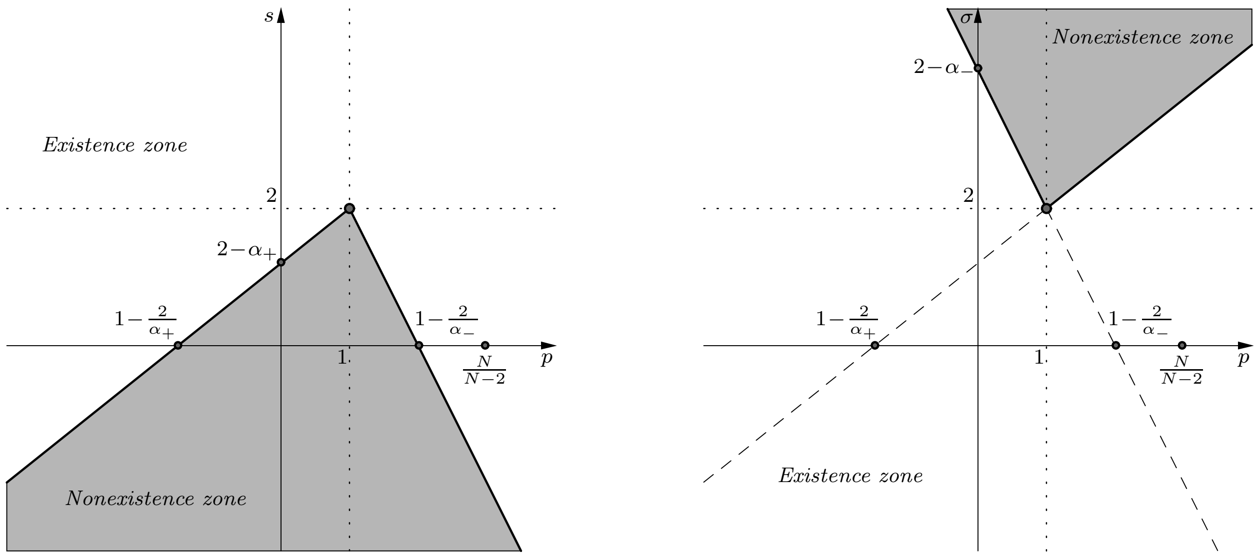

Figure 1: Existence and nonexistence zones for equations (1) (left) and (2) (right).

where $\sigma = (N+2) - p(N-2) - s$ and $\widehat{\mathcal{C}}^1_\Omega := \{(r,\omega) \in \mathbb{R}^N : \omega \in \Omega,\ 0 < r < 1\}$. We define the critical exponents $\widehat{p}^* = \widehat{p}^*(\Omega, s)$ and $\widehat{p}_* = \widehat{p}_*(\Omega, s)$ for equation (2) similarly to $p^*(\Omega, s)$ and $p_*(\Omega, s)$. In the superlinear case $p > 1$, Bandle and Essen [2] proved that if $\sigma > 2$ then $\widehat{p}^* = 1 - \frac{2-\sigma}{\alpha_+}$ and (2) has no positive supersolutions when $p = \widehat{p}^*(\Omega)$. In the sublinear case $p < 1$ by an easy computation we derive from Theorem 1 the following result.

Theorem 2. *For $p \le 1$, the critical exponent for equation (2) is $\widehat{p}_* = \min\{1 - \frac{2-\sigma}{\alpha_-}, 1\}$. If $\widehat{p}_* < 1$ then (2) has no positive supersolutions in the critical case $p = \widehat{p}_*$.*

In the remaining part of the paper we prove Theorem 1.

2 Proof of Theorem 1

Existence. In the polar coordinates equation (1) reads as follows

$$-u_{rr} - \frac{N-1}{r}u_r - \frac{1}{r^2}\Delta_\omega u = \frac{c}{r^s}u^p \quad \text{in } \mathcal{C}^1_\Omega. \tag{3}$$

Let $s \le 2$, $p < 1 - \frac{2-s}{\alpha_+}$. Let $0 < \psi \in H^1_{loc}(\Omega)$ be a positive solution to the equation

$$-\Delta_\omega \psi - \alpha(\alpha + N - 2)\psi = \psi^p \quad \text{in } \Omega, \tag{4}$$

where $\alpha := \frac{2-s}{1-p}$. Then it is readily seen that $u := c^{\frac{1}{1-p}} r^\alpha \psi \in H^1_{loc}(\mathcal{C}^1_\Omega)$ is a positive solution to (3) in $\mathcal{C}^1_\Omega$. Thus the problem reduces to the existence of positive solutions to (4).

Note that $0 < \alpha(\alpha + N - 2) < \lambda_1(\Omega)$. Hence the operator $-\Delta_\omega - \alpha(\alpha + N - 2)$ is coercive on $H^1_0(\Omega)$ and satisfies the maximum principle. We consider separately the cases $p \in [0, 1)$ and $p < 0$.

Case $p \in [0,1)$. Let $\phi_1 > 0$ be the principal Dirichlet eigenfunction of $-\Delta_\omega$ on Ω. Let $\overline{\phi} > 0$ be the unique solution to the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)\phi = 1, \qquad \phi \in H_0^1(\Omega).$$

Observe that $\phi_1, \overline{\phi} \in L^\infty$.

Hence $\tau\overline{\phi}$ is a supersolution to (4) for a large $\tau > 0$, and $\epsilon\phi_1$ is a subsolution to (4) for a small $\epsilon > 0$. Thus by the sub and supersolutions argument equation (4) has a solution $\psi \in H_0^1(\Omega)$ such that $\epsilon\phi_1 < \psi \leq \tau\overline{\phi}$.

Case $p < 0$. Consider the problem

$$-\Delta_\omega \phi - \alpha(\alpha + N - 2)(\phi + 1) = (\phi + 1)^p, \qquad \phi \in H_0^1(\Omega). \tag{5}$$

Let $\overline{\phi} > 0$ be the unique solution to the problem

$$-\Delta \phi - \alpha(\alpha + N - 2)(\phi + 1) = 1, \qquad \phi \in H_0^1(\Omega).$$

It is clear that $\overline{\phi}$ is a supersolution to (5) and $\underline{\phi} \equiv 0$ is a subsolution to (5). We conclude that (5) has a positive solution $\phi \in H_0^1(\Omega)$ such that $0 < \phi \leq \overline{\phi}$. Then $\psi := \phi + 1 \in H_{loc}^1(\Omega)$ is a positive solution to (4). This completes the proof of the existence part of Theorem 1.

Nonexistence. In what follows we set $\delta := 1$ if $p < 0$ and $\delta := 0$ if $p \in [0,1)$. Let $G \subset \mathbb{R}^N$ be a domain, $0 \notin G$. Observe that equation (1) has a positive supersolution in G if and only if the equation

$$-\Delta w = \frac{c}{|x|^s}(w + \delta)^p \quad \text{in } G \tag{6}$$

has a positive supersolution. Indeed, if $u > 0$ is a supersolution to (1) in G then u is a supersolution to (6). If $w > 0$ is a supersolution to (6) then $u = w + \delta$ is a supersolution to (1). The main argument of the proof nonexistence rests upon the following two lemmas.

The next lemma is an adaptation a comparison principle by Ambrosetti, Brezis and Cerami [1, Lemma 3.3].

Lemma 3. *Let $G \subset \mathbb{R}^N$ be a bounded domain, $0 \notin G$. Let $0 \leq \underline{w} \in H_0^1(G)$ be a subsolution and $0 \leq \overline{w} \in H_{loc}^1(G)$ a supersolution to (6). Then $\underline{w} \leq \overline{w}$ in G.*

Proof. In [1, Lemma 3.3] the result was proved for a smooth bounded domain G and $\underline{w}, \overline{w} \in H_0^1(G)$ (and more general nonlinearities). The proof given in [1] carries over literally to the case of an arbitrary bounded domain G and $\underline{w}, \overline{w} \in H_0^1(G)$, or a smooth bounded domain G, $\underline{w} \in H_0^1(G)$ and $0 \leq \overline{w} \in H^1(G)$. Thus we only need to extend the lemma to an arbitrary bounded domain G and $\overline{w} \in H_{loc}^1(G)$.

Let $\overline{w} \in H_{loc}^1(G)$ be a supersolution to (6) in G. Let $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G, that is a sequence of bounded smooth domains such that $\overline{G}_n \subset G_{n+1} \subset G$ and $\cup_{n\in\mathbb{N}} G_n = G$. Analogously to the argument given above in the existence part of the proof, one can readily see that, for each $n \in \mathbb{N}$, there exists a solution $0 < w_n \in H_0^1(G_n)$ to (6) (e.g., by constructing appropriate sub and supersolutions). Moreover, $w_n \leq w_{n+1}$. Observe that $w_n \leq \overline{w}$ in G_n by [1, Lemma 3.3].

We claim that $\sup \|\nabla w_n\|_{L^2} < \infty$. This is clear for $p < 0$, since $(w+1)^p \leq 1$. For $p \in [0,1)$, using w_n as a test function in (6), we have

$$\int_G |\nabla w_n|^2 dx = \int_G \frac{c}{|x|^s} w_n^{p+1}\, dx \leq c_1 \left(\int_G |\nabla w_n|^2 dx \right)^{(p+1)/2},$$

which implies the claim. It follows that w_n converges pointwise in G, strongly in $L^2(G)$ and weakly in $H_0^1(G)$ to a positive $w_* \in H_0^1(G)$. Clearly $w_* > 0$ is a solution to (6) in G and $0 < w_* \leq \overline{w}$ in G.

Now let $0 \leq \underline{w} \in H_0^1(G)$ be a subsolution to (6) in G. By [1, Lemma 3.3] we conclude that $\underline{w} \leq w_*$ in G. $\square$

Next, consider the initial value problem

$$(7) \qquad -v_{rr} - \frac{N-1}{r} v_r + \frac{\lambda_1}{r^2} v = \frac{c}{r^s} v^p \quad \text{for } r > 1; \qquad v(1) = \delta, \quad v_r(1) = K;$$

where $p < 1$, $s \in \mathbb{R}$, $c > 0$, $K > 1$ and δ as above. Let $(1, R)$, $R = R(\delta, K) \leq \infty$, be the maximal right interval of existence of the solution v to (7) in the region $\{(r,v) \in (1,+\infty) \times (\delta, +\infty)\}$.

Lemma 4. *Let $s < 2$ and $p \in [1 - \frac{2-s}{\alpha_+}, 1)$. Then for any interval $[r_*, r^*] \subset (1, +\infty)$ there exists $K_0 > 1$ such that*

i) for all $K > K_0$ one has $r^* < R < +\infty$ and $v(r) \to \delta$ as $r \nearrow R$;

ii) for any $M > \delta$ there exists $K > K_0$ such that $\min_{[r_*, r^*]} v \geq M$.

Proof. Set $\alpha := \alpha_+$, $v := w r^\alpha$, $t = r^{2-N-2\alpha}$. Then w solves the following problem

$$w_{tt} + c_1 t^{-\sigma} w^p = 0 \quad \text{for } t \in (T, 1); \qquad w(1) = \delta, \quad w_t(1) = -L,$$

where $\sigma = \frac{2N-2+\alpha(p+3)-s}{N-2+2\alpha} \geq 2$, $c_1 > 0$, $0 \leq T = R^{2-N-2\alpha} < 1$ and $L = \frac{K - \alpha\delta}{N-2+2\alpha} \to \infty$ as $K \to \infty$. Choose K_0 such that $L > \delta$. Observe that $w(t)$ is concave, hence

$$\delta < w(t) \leq w(1) - w_t(1)(1-t) \leq \delta + L \quad \text{for } t \in (T, 1).$$

To see that $T > 0$ let $\tilde{w} := w$ for $p < 0$, otherwise let $\tilde{w} := w^{1-p}$. Then $\tilde{w}$ satisfies the inequality

$$\tilde{w}_{tt} + c_2 t^{-2} \tilde{w}^q \leq 0 \quad \text{for } t \in (T, 1),$$

with $c_2 > 0$ and $q := \min\{p, 0\}$. Integrating $\tilde{w}_{tt}$ twice one can easily see that such inequality has no positive solutions in any neighborhood of zero. Thus we conclude that $T > 0$, hence $w(t) \to \delta$ as $t \searrow T$. In particular, $w(t)$ attains its maximum on $(T, 1)$.

Let $T_0 \in (T, 1)$ be such that $w_t(T_0) = -\frac{L-\delta}{2}$. Since $\delta \leq w(t) \leq \delta + L$ for $t \in (T_0, 1)$, it follows that

$$\frac{L+\delta}{2} = w_t(T_0) - w_t(1) = -\int_{T_0}^1 w_{tt} d\tau = c_1 \int_{T_0}^1 \frac{w^p}{\tau^\sigma} d\tau \leq c_3 \left(\frac{1}{T_0^{\sigma-1}} - 1 \right) \quad \text{for } t \in (T_0, 1).$$

Hence $T_0 \to 0$ as $L \to +\infty$. Therefore for any given $t^* < 1$ there exists $L_0 > 1$ such that for any $L > L_0$ one has $0 < T < T_0 < t^*$. Thus, (i) follows with $r^* = (t^*)^{\frac{1}{N-2+2\alpha}}$.

Observe now that for any $L > L_0$ we have

$$-\frac{L-\delta}{2} \geq w_t(t) \geq -L \quad \text{for } \ t \in (t^*, 1),$$

since w is concave. Hence for any $t \in (t^*, 1)$ we obtain

$$w(t) = w(1) - \int_t^1 w_t \, d\tau \geq \delta + (1-t)\frac{L-\delta}{2} \to \infty \quad \text{as } \ L \to \infty.$$

Thus (ii) follows. $\square$

Nonexistence – completed. Let $p \in [1 - \frac{2-s}{\alpha_+}, 1)$. Fix a compact $K \subset \mathcal{C}^1_\Omega$ and $M > 1$. There exists an interval $[r_*, r^*] \subset (1, +\infty)$ such that $K \subset \mathcal{C}_\Omega^{(r_*, r^*)}$, where $\mathcal{C}_\Omega^{(r_1, r_2)}$ denotes the set $\{x \in \mathcal{C}^1_\Omega \,|\, r_1 \leq |x| \leq r_2\}$. Then by Lemma 4 there exists $v : (1, R) \to (\delta, +\infty)$ solving (7) such that $R > r^*$ and $\inf_{[r_*, r^*]} v \geq M + \delta$.

Let $\phi_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta_\omega$ on Ω with $\|\phi_1\|_\infty = 1$. Set $w_M := (v - \delta)\phi_1$. Then $0 < w_M \in H^1_0(\mathcal{C}_\Omega^{(1,R)})$, and direct computation shows that w_M is a subsolution to (6) in $\mathcal{C}_\Omega^{(1,R)}$. Now assume that $w > 0$ is a supersolution to (6) in $\mathcal{C}^1_\Omega$. By Lemma 3 it follows that that $w \geq w_M$ in $\mathcal{C}_\Omega^{(1,R)}$. By the weak Harnack inequality we have

$$\inf_K w \geq c_H \int_K w \, dx \geq c_H \int_K w_M \, dx \geq c_2 M.$$

Since M was arbitrary, we conclude that $w \equiv +\infty$ in K. $\square$

Acknowledgement. The support of the MFI Oberwolfach, Nuffield Foundation and Institute of Advanced Studies of Bristol University is gratefully acknowledged.

References

[1] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effect of concave and convex nonlinearities in some elliptic problems.* J. Funct. Anal. **122** (1994), 519–543.

[2] C. Bandle and M. Essén, *On positive solutions of Emden equations in cone-like domains*, Arch. Rational Mech. Anal. **112** (1990), 319–338.

[3] C. Bandle and H. A. Levine, *On the existence and nonexistence of global solutions of reaction–diffusion equations in sectorial domains.* Trans. Amer. Math. Soc. **316** (1989), 595–622.

[4] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems.* Topol. Methods Nonlinear Anal. **4** (1994), 59–78.

[5] H. Brezis and S. Kamin, *Sublinear elliptic equations in* $\mathbb{R}^N$, Manuscripta Math. **74** (1992), 87–106.

[6] V. Kondratiev, V. Liskevich and V. Moroz, *Positive solutions to superlinear second–order divergence type elliptic equations in cone–like domains*, Preprint, 2003.

[7] V. Kondratiev, V. Liskevich and Z. Sobol, *Second–order semilinear elliptic inequalities in exterior domains*, J. Differential Equations **187** (2003), 429–455.